ACTA CHEMICA SCANDINAVICA 12 (1958) 415—437

Servo-controlled Wheatstone Bridge Arrangements for the Simultaneous Recording of Resistance and Capacitance

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The conditions for simultaneous servo-balancing of resistance and capacitance in alternating-current Wheatstone bridges of various types are discussed. A series of experiments are described which confirm the theory, and give information regarding the practical aspects of double-servo-controlled A.C. bridges.

Servo-controlled Wheatstone bridges intended for the continuous recording of resistance in electrolytic systems have been described in the literature. For references see Ref.[1] The arrangements so far described use servo control only for the resistive balance, and the capacitative balance has to be adjusted manually. It was suggested by Andersson, Mellander, and Stenhagen [1] that automatic capacitative correction might be used in the form of a standardization operation, performed in a manner similar to the automatic current standardization in D.C. millivolt recorders of, for example, the "Speedomax" type. A study of the theoretical side of the problem, however, revealed that it should be possible to balance both capacitance and resistance simultaneously by using two independent servo systems, and the results recorded in the experimental part of this communication show that servo-controlled Wheatstone bridge arrangements for the simultaneous recording of resistance and capacitance are capable of practical realization.

THEORY. CONDITIONS FOR OPERATION OF DOUBLE SERVO SYSTEM

I. Normal Wheatstone bridge (Fig. 1)

Consider the Wheatstone bridge arrangement of Fig. 1. It is assumed throughout this communication that the fixed bridge arms AD and DC are identical. The variable impedance to be recorded (for instance that of an electrolytic cell) is connected between points B and C, and considered as a

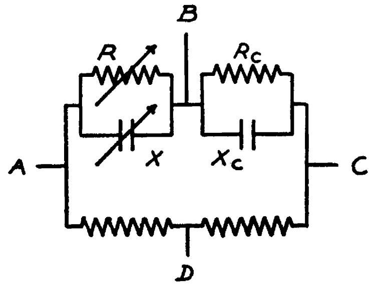

Fig. 1. Normal alternating current bridge. A and C are connected to a source of alternating current; B and D are connected to the detector. Shielding and Wagner earthing arrangements not shown.

resistance and a capacitance in parallel. A source of A.C. current is connected across points A and C. When the bridge is unbalanced, that is, when $R \neq R_C$ and/or $X \neq X_C$, an alternating potential exists across BD. The unbalance is to be automatically corrected through the variable resistance R and reactance X, each controlled by its own servo-motor. The servos are steered by the potential across points B and D, through phase-sensitive detectors. The sign of the D.C. output of a phase sensitive detector depends on the phase difference between the signal and a reference potential of the same frequency (in the present case obtained from the oscillator supplying the bridge current). If the phase difference is $< 90°$, the detector gives a D.C. component in the output current; and if the phase difference is $> 90°$, the D.C. component has the opposite sign. If the phase difference is exactly 90° (or 270°) there will be no D.C. component in the output.

We now connect two phase-sensitive detectors in parallel across BD (in practice they are connected in parallel over the output of the detector amplifier), and the D.C. outputs are fed to the R- and the X-servo, respectively. The reference potentials are termed $V_{R\,ref.}$ and $V_{X\,ref.}$

If the phase of the alternating potential V_{DB} is not exactly 90° out of phase with respect to $V_{X\,ref.}$, the X-servo receives a signal, and will adjust the reactance X until the signal output of the X-detector is zero (or is below the threshold of the servo). The R-servo will operate in a corresponding manner. If the $V_{R\,ref.}$ and $V_{X\,ref.}$ are out of phase with respect to each other,

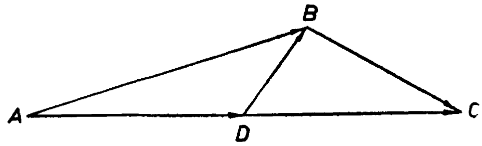

Fig. 2. Vector diagram of the potentials of the bridge shown in Fig. 1 at unbalance.

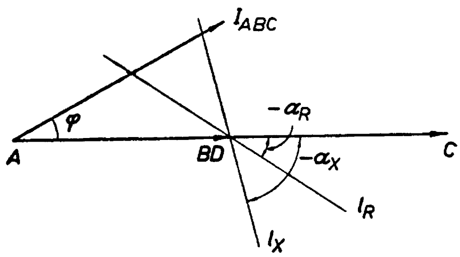

Fig. 3. Current and potential vectors of the bridge shown in Fig. 1 at balance. For the definition of l_X and l_R see text.

it is evident that V_{DB} must be zero in order that neither servo shall operate, which means that the desired balance of the bridge has been attained. The servos may be coupled to recorders that automatically record both the resistance and the reactance of the bridge-arm BC.

In order for the arrangement to function properly, two conditions must be fulfilled: (a) "Coupling" between the servos must be avoided, that is, *if one of the servos receives no signal the other servo should not be able to alter this condition;* and (b) "Parallelism" between a servo and its steering function must be avoided, that is, *if one of the servos receives a signal it must be possible for the servo in question to alter this condition.*

It should be possible to realize condition (a) approximately and condition (b) (which is the more important) exactly, by a suitable choice of phase for the reference signals. Fig. 2 shows in vector form the potentials between the four corners of the bridge at unbalance. The vector representing the potential V_{DB} can form any angle with the vector V_{AC}, but is always small compared with the latter. At balance $V_{DB} = 0$: B coincides with D as shown in Fig. 3. If $R = R_C$, and R, R_C, and X_C are kept constant, and X varies (*cf.* Fig. 1), the point B of the vector V_{DB} of Fig. 3 will move along a curve. If X is kept constant and X_C varies, point B will move along a different curve. In point D these curves have the common tangent l_X, forming an angle α_X with the vector V_{AC}. Analogous relations are valid for R, R_C, and α_R of Figs. 1 and 3. If V_{DB} is kept small we may use the approximation that the point B moves along the tangents l_X and l_R. For the purpose of our discussion we can furthermore assume that the input resistance of the detector amplifier is infinitely high, and that therefore no current flows between points B and D. Any current I_{DB} will cause changes in all potentials, and probably also in α_R and α_X. If $I_{DB} = 0$ it can be shown (see Appendix I) that $\alpha_R = -\varphi$ and $\alpha_X = -(\varphi + 90°)$, where φ is the angle between V_{AC} and I_{ABC}. It follows that if, for example, R_C changes during an experiment, and R (to keep the balance) changes in the same direction, φ and consequently also α_R and α_X will change. In the following discussion, however, we may disregard this change.

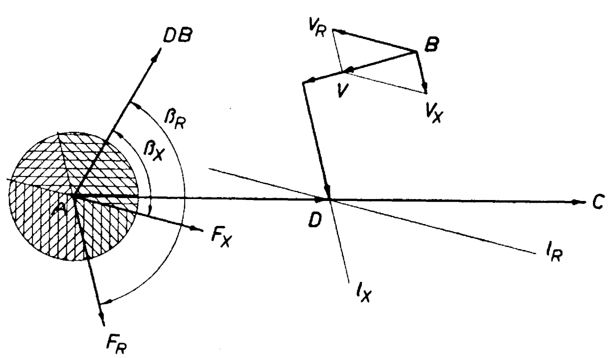

Fig. 4. Diagram illustrating the servo balancing of the bridge shown in Fig. 1.

In order to avoid coupling between the servos, the X-servo must not operate when point B is moving along l_R, and B therefore can be led back to D with the R-servo alone. The X-servo does not operate if the potential vector $V_{X\ ref.}$ is normal to l_R. Analogously, $V_{R\ ref.}$ should be normal to l_X. We then get the conditions shown in Fig. 4. The sign of the output signals from the phase detectors are determined by the angles β_R and β_X between the vector V_{DB} and the vectors F_X and F_R, respectively, which are normals to the corresponding reference potentials. When $\beta_R > 0$ (region \\\\\\) the R-servo will cause point B to move with velocity v_R towards "west-north-west" along a line approximately parallel to l_R. If $\beta_R < 0$ (region //////), v_R will point towards "east-south-east" instead. Similarly, $\beta_X > 0$ (region ≡≡≡) gives v_X pointing towards "south-south-east", approximately parallel to l_X; and $\beta_X < 0$ (region ||||||) gives v_X pointing in the opposite direction ("north-north-west"). In the circumstances illustrated in Fig. 4, both β_R and β_X are > 0 (region ▦), and v_X and v_R add vectorially to v, pointing towards "west-south-west" until point B reaches the line l_X, when the R-servo stops. Only the component v_X of v then remains ($\beta_R = \pm 180°$, but β_X still > 0), and point B moves along l_X until point D is reached. The X-servo then stops, and the bridge is balanced.

A block diagram of the suggested servo-controlled bridge arrangement with independent R- and X-servos is shown in Fig. 5. The adjustment of the phase of the reference signals with the aid of the two phase shifters may be carried out as follows. The electrolytic cell or other impedance to be recorded is connected to the bridge. The phase detectors are disconnected and the bridge balanced manually in the usual manner. The phase detectors are connected to the bridge amplifier and the balance of the bridge slightly altered by altering R only, making point B in the diagram of Fig. 3 move out from D along l_R. This will probably result in output signals from both phase detectors. However, we wish only to get an output signal form the R-detector when R is changed and therefore adjust the phase of the reference signal $V_{X\ ref.}$ of

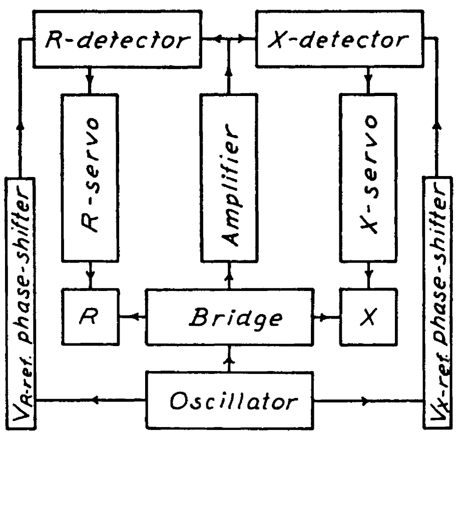

Fig. 5. Block diagram of bridge arrangement with double servo.

the X-detector until no output signal is obtained from this detector. $V_{X\ ref.}$ is then parallel to l_R. The bridge is again balanced manually, and the procedure just described repeated with reactance X and the phase shifter for $V_{R\ ref.}$ The servos are then connected. It may be necessary to shift the polarity of the servo connection if the phase of the corresponding reference signal should happen to be shifted 180°. The apparatus should now be ready for the run. For practical details see the experimental part.

A slight coupling between the servos must be tolerated for the following reasons. (a) The straight lines l_R and l_X are only approximations to the curves along which point B (Fig. 3) moves. (b) As already pointed out, the angles α_R and α_X of Fig. 3 vary with the phase-angle φ between the potential V_{AC} and the current I_{ABC}. This means that the orthogonality between the reference signals and the l-lines will not be exact. The errors can be kept small, however, provided that (a) point B is not allowed to move far away from D, that is, the servos must keep up well, and (b) the phase-angle does not undergo large absolute changes during the run. Difficulties due to phase-angle variation may be overcome by using the series bridge (see III).

II. The "potentiometer" Wheatstone bridge arrangement

The "potentiometer" bridge arrangement is extremely convenient for practical measurements of resistance, as it allows a continuous variation of the recorder range by means of a shunt across the potentiometer. In the apparatus described previously [1], the recorder range could be varied from 2 to 1 800 ohms full scale. The bridge is shown in Fig. 7. The use of a potentiometer in this manner may sometimes lead to errors in the recordings, however. Although the presence of such errors may be checked experimentally, a theoretical analysis is warranted.

As the complicated impedance of the arm BC (Fig. 7) makes theoretical calculations very difficult, the arrangement shown in Fig. 8 is used instead. The "differential" condenser (ganged X_1, X_2) is replaced by two separate condensers connected across A—B_a and C—B_c, respectively. Only X_2 should be

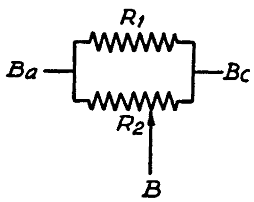

Fig. 6. Diagram showing shunted potentiometer.

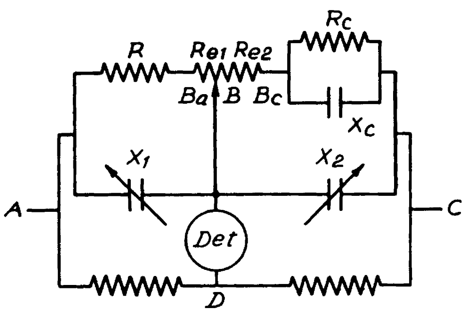

Fig. 7. Potentiometer bridge described in Ref.[1], and used in some of the experiments in Section VI.

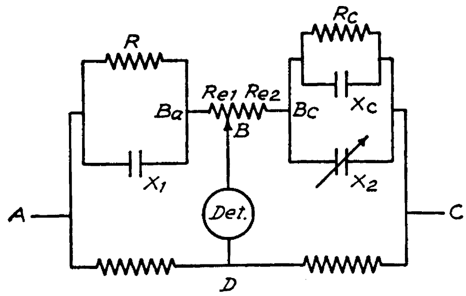

Fig. 8. Bridge arrangements recommended instead of that shown in Fig. 7.

servo-operated, as X_1, not being connected across exactly the same resistance as X_C (R is kept constant but the cell resistance R_C varies during the experiment), does not have the same action as X_C on the circuit. X_1 and X_2 should be as large as possible in order to keep φ small, because, as will be shown, a large φ causes poor linearity and poor servo action.

In order to simplify the calculations the shunted potentiometer (Fig. 6.) is replaced by the equivalent resistance R_e. The potentials B_aB and BB_c depend only on the position of B along R_1 and the total potential B_aB_c and this will be the same if the combination R_1, R_2 is replaced by the equivalent resistance

$$R_e = R_1R_2/(R_1 + R_2).$$

In the bridge arrangement of Fig. 1 it is evident that R is equivalent to R_C, and X to X_C. In the potential vector diagram given in Fig. 3 we therefore have only two "tracks" for the vector arrow point B, *viz.*, l_R and l_X. With the circuit of Fig. 8 this is no longer the case, and from the outset we have to consider four tracks, l_{R_C}, l_{X_C}, l_{R_e}, and l_{X_2}. We need to know the answer to the following questions.

(a) Is it possible to compensate a small change in R_C with R_e only? This is of importance, particularly in the measurement of X_C. If the answer is no, it means that the condenser servo (X_2) responds both to capacitative and resistive changes of the cell, and it may be difficult to derive the capacitative changes from the recordings. This question, which is less important for the measurement of R_C, may also be formulated as follows: Do l_{R_C} and l_{R_e} coincide?

(b) Do l_{X_C} and l_{X_2} coincide? This question is of particular importance in the measurement of R_C.

(c) What are the quantitative relations between R_C and R_e?

(d) What are the quantitative relations between X_C and X_2?

It is evident that question (b) can be answered in the affirmative, because X_C and X_2 are connected in parallel and have the same influence in the circuit. Resistance measurements are thus possible. Calculations, the details of which are given in Appendix II, show that the relation between ΔR_e and ΔR_C is

$$\Delta R_e = \frac{1}{1 - \mathrm{tg}^2\,\varphi} \cdot \tfrac{1}{2}\,\Delta R_C \qquad (14)$$

If R_e is mechanically coupled to a recorder we get a recording of ΔR_C in accordance with eqn. (14). The linearity is good, provided that φ is small and does not vary too much. If $\text{tg}^2\varphi$ is less than 0.003 ($0 < \varphi < 2.75°$) ΔR_e will differ from $\frac{1}{2}\Delta R_C$ by less than 0.3 %. For phase angles smaller than the value just given, the error is thus less than the writing error of precision recorders of the "Speedomax" type. As tg φ increases with increasing frequency, the error due to the shunted potentiometer circuit will be smaller with a lowering of the frequency of the bridge current. On the other hand, the time constants of the filters necessary in the detector circuits have to be increased with a lowering of the frequency, leading to a fall in the speed of response on the part of the servos. The relation between the susceptance, $B_2 = 1/X_2$, of the condenser X_2 and R_C and X_C may be expressed by the following equations (for derivation see Appendix II).

$$\Delta B_2 = -\frac{2 \text{ tg } \varphi}{R_C^2 (1 - \text{tg}^2\varphi)} \cdot \Delta R_C \tag{13}$$

$$\Delta B_2 = -\Delta \frac{1}{X_C} \tag{15}$$

If we couple a recorder mechanically to the condenser X_2, the recorder will write a curve of the sum of a susceptance curve according to eqn. (15), and a resistance curve according to eqn. (13). From Fig. 9 we obtain the value $90° - 2\varphi_2 \approx 90° - 2\varphi$ for the angle between l_{R_e} and l_X. When $\varphi = 45°$, l_{R_e} and l_X coincide. The action of the potentiometer will then be the same as that of condenser X_2, and it will be impossible to balance the bridge. It may perhaps be possible to allow values of φ as high as 30°, but it must be kept in mind that the permitted variation in φ will become less with increasing φ.

To sum up, the "potentiometer" bridge circuit is unsuitable for the recording of reactance unless tg φ is very close to zero (eqn. 13), but it may be employed for the recording of the resistive component of an impedance that does not vary over large ranges.

III. The series bridge

For the measurement of electrolytic resistance, bridges with parallel combinations of resistances and capacitances are usually employed. However, a

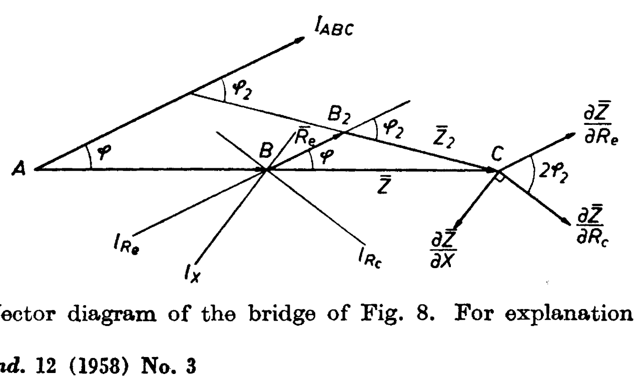

Fig. 9. Vector diagram of the bridge of Fig. 8. For explanation see text.

series bridge arrangement has two very important advantages over the parallel bridge with respect to servo control: (a) It is possible to use a shunted, servo-controlled measuring potentiometer without introducing any theoretical error; and (b) The line l_R of the potential diagram (Fig. 11) is parallel, and the line l_X normal, to I_{ABC}. If the reference potentials of the detectors are based upon the current I_{ABC}, they will retain their theoretically correct value independent of variations in φ; and if the servos follow well, there will never be any coupling between them.

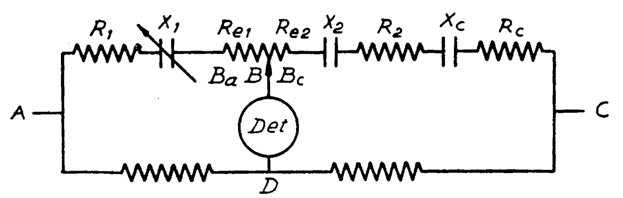

Fig. 10. Series bridge.

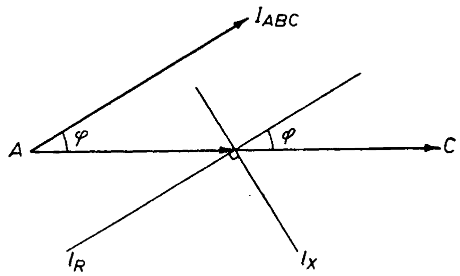

Fig. 11. Vector diagram of the series bridge.

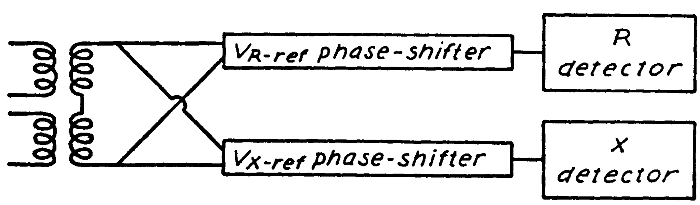

Fig. 12. Suggested detector arrangement for the series bridge.

The bridge may be connected as shown in Fig. 10. The series bridge is particularly useful when measuring impedances with large variations in φ. The reference potentials may be obtained through a suitable transformer with two identical primary windings (Fig. 12) connected in series with X_1 and X_2. An amplifier may have to be used between the transformers and the phase shifters.

In order to measure with the series bridge the parallel components of, say, an electrolytic cell, it is necessary to determine both series components and to calculate the parallel components of the cell using the following expressions

$$R_p = R_s (1 + R_s^2 B_s^2) \text{ and } X_p = \frac{R_s^2 B_s^2 + 1}{R_s B_s^2}.$$

IV. The Easton-Lamson bridge (General Radio Z-Y-bridge)

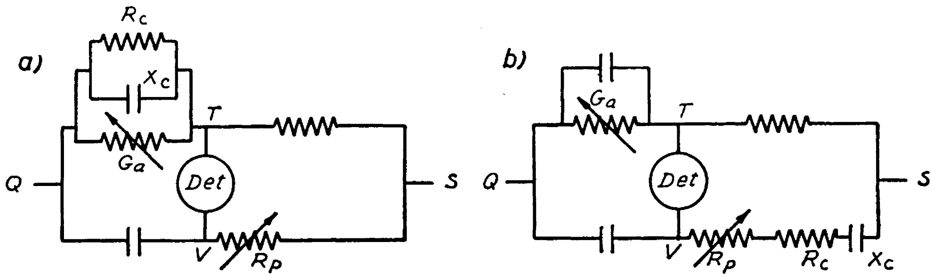

Fig. 13. a) Z—Y bridge with "cell" measured as admittance. b) Z—Y bridge with "cell" measured as impedance.

Easton and Lamson [2] have recently described the bridge arrangement shown in Fig. 13. This bridge has the distinction that it can be balanced for any type of impedance to be measured, and possesses over the normal Wheatstone bridge of Fig. 1 the advantage that both reactance and resistance are balanced with purely resistive components. It is therefore possible to use two similar resistances, one coupled to a recorder and the other to a simple servo, and simply change the connections according to whether the resistance or the reactance is to be recorded.

Fig. 14 shows the potential diagram at unbalance. When the bridge is balanced, points T and V (corresponding to points B and D of the diagram of the normal bridge of Fig. 2) coincide, as in Fig. 15. The resistance G_a moves

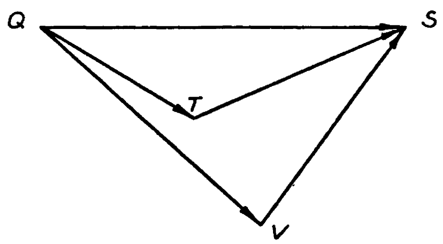

Fig. 14. Vector diagram of Z—Y bridge at unbalance.

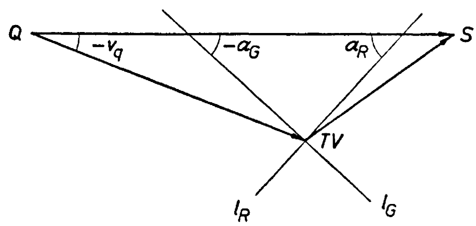

Fig. 15. Vector diagram of Z—Y bridge at balance. For explanation of l_R and l_G see text.

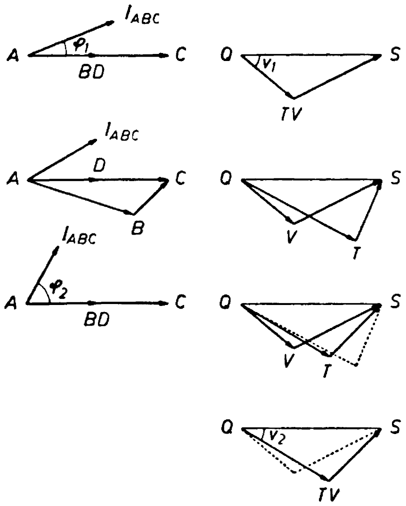

Fig. 16. Vector diagrams showing the balancing of the normal bridge left (1. Bridge balanced. 2. Unbalance introduced. 3. Balance restored with R and X.) and the Z-Y-bridge right (1. Bridge balanced. 2. Unbalance introduced. 3. T moved to balance point by means of G_a. 4. Balance restored by means of R_p.)

the point T along the line l_G, and R_p moves V along l_R. If the "cell" is measured as an "admittance", l_G and l_R correspond to l_R and l_X, respectively, in Fig. 3. The angles α_G and α_R that the lines l_G and l_R form with the impressed voltage QS are derived in appendix III. They are

$$\alpha_G = 2v_q \qquad (16)$$

$$\alpha_R = 2v_q + 90^\circ \qquad (17)$$

and obviously depend only on the angle v_q between the voltages QS and QT (or QV, which is similar to QT), in the same way as α_R and α_X in Fig. 3 depend on the phase angle φ. And, unfortunately, like α, v_q cannot always be kept constant during the run.

It is clear, however, that in principle it is possible to servo-control the ZY-bridge in the same way as the normal bridge. The series of figures above (Fig. 16) shows what happens when the balance is disturbed and restored in the two cases. The impedance to be measured can be connected to the ZY-bridge in two different ways (Fig. 13). If connected as an "impedance" (Fig. 13 b) it is measured as a resistance and a reactance in series; and if it is connected as an "admittance" (Fig. 13 a) it is measured as a resistance and a reactance in parallel. In the former case the series reactance is not given by G_a directly, because the curve will give the variation of $\Delta\ 1/G_a$ and not $\Delta\ G_a$ if a linear resistance is used. (In the bridge manufactured by the General Radio Co. the measuring resistance is linear in conductance, not in resistance.) This is usually inconvenient, and it is therefore better to connect the unknown impedance according to the second alternative, and measure it as an admittance. If the connected admittance should fall outside the range of the instrument, the parameters of the bridge arms must be changed. When measuring the unknown impedance as admittance, G_a writes a curve giving the parallel resistance, and R_p a curve of the parallel susceptance.

V. Concerning the possibility of using a Wheatstone bridge with servo, without manual or servo-control of the quadrature component

Two cases are at once apparent when a single servo is sufficient, *viz.* when φ is constant and equal to zero, and when φ is constant and equal to 90°. It may be readily assumed that it should also be possible to use a single servo, when, for instance, it is desired to measure changes in resistance in cases where the capacitance remains constant (but $\neq 0$). This is in fact possible if the normal bridge arrangement of Fig. 1 is used, but is in general less satisfactory in case of the "potentiometer" circuit (Figs. 6 and 8), because, as already discussed above, a change in R_C causes both resistive and capacitative unbalance. However, if it is desired to measure only small changes in one of the parallel components, is it then necessary to keep the bridge in balance with respect to both components? A reference to Fig. 4 shows that if φ is constant, and if changes in R always make B move along straight lines parallel to l_R independent of the position of B in the plane, and if corresponding conditions are valid for l_X and X, X may remain in any position and X_C vary at will if R_C is to be measured. The R-servo and the attached recorder ought to record the same curve as when the bridge is kept in complete balance. Instead of remaining at D, the point B would move up and down along l_X, but this would have no influence on the R-servo.

Conditions are of course less perfect in practice, but the smaller the variation in Z, the nearer will be the approach to the ideal. The changes in Z to be recorded are often small compared with Z. It is, however, essential that the phase angle of the reference potential is correctly chosen.

Finally, it must be pointed out that the above reasoning implies that the detector does not lose its sensitivity when a large signal is applied, that is to say, a "logarithmic" detector amplifier cannot be used.

VI. Experimental part

The bridge and associated apparatus were the same as used previously [1], with the addition of a capacitance servo consisting of phase-sensitive detector with built in phase shifter (identical to that used for the recording of resistance) and a "Speedomax" recorder with a writing-speed of 2.5 seconds across the scale. A precision-built air-dielectric condenser * of 500 pF capacitance with linear capacitance variation, contained in a shielded box attached to the back of the recorder housing, was mechanically coupled to the recorder potentiometer shaft via an insulated bushing. A D.C. tachometer generator was attached to the recorder motor as described in the previous paper [1]. In Ref.[1] the bridge was arranged as shown in Fig. 7, but in the first series of experiments to be described below the bridge was set up as in Fig. 1, by joining points B and Bc. The servo-controlled condenser was connected between A and B. The servo was controlled by the detector signal across BD (Fig. 1) *via* the same detector amplifier ** with automatic volume control as the resistance servo, the two phase-sensitive detectors being connected in parallel across the output.

* The condenser used was of an early radio-receiver type manufactured by AB Baltic, Södertälje.

** The time constant of the A.V.C. action is about 0.2 seconds, and not 2 seconds as was erroneously given in Ref.[1]

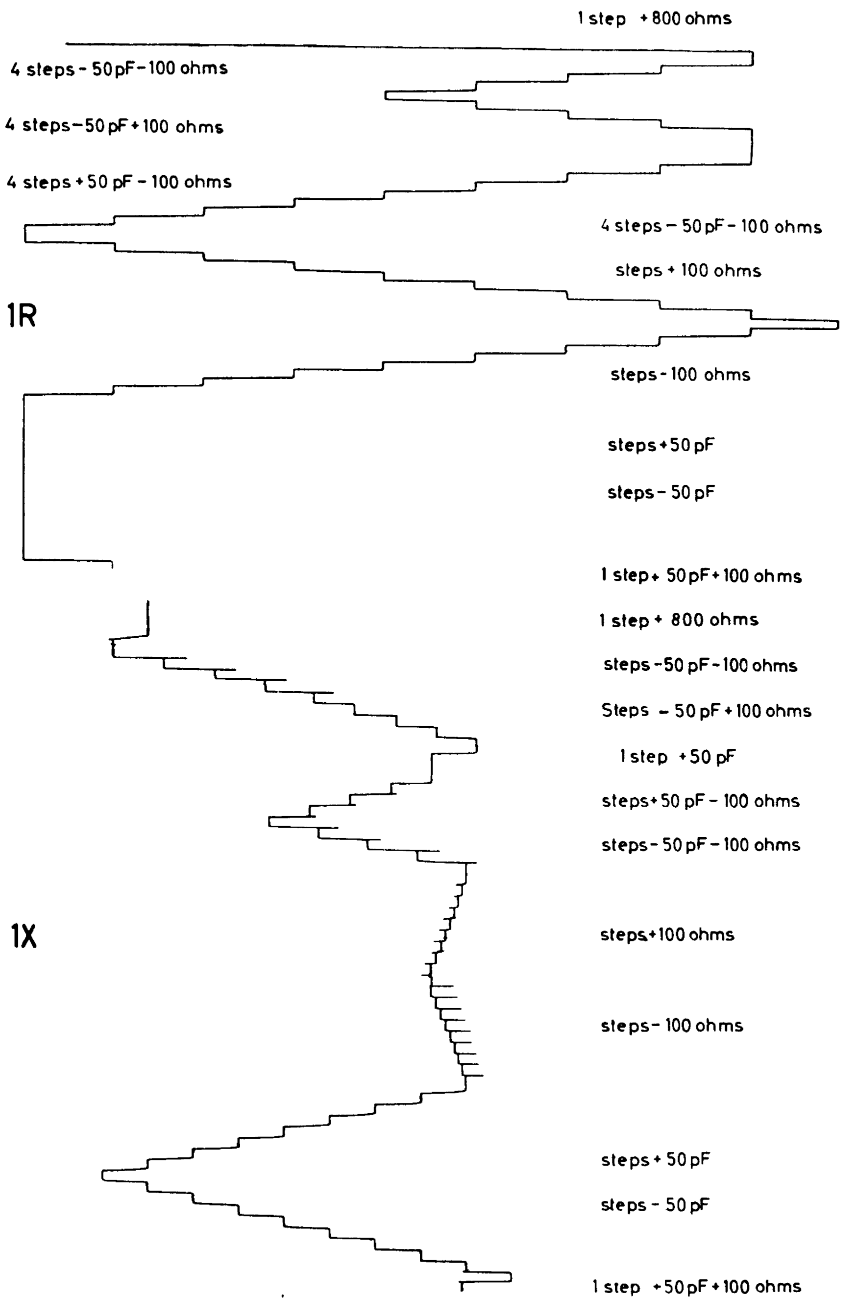

Fig. 17. Double servo-recordings using the bridge shown in Fig. 1. Sinusoidal reference potentials.

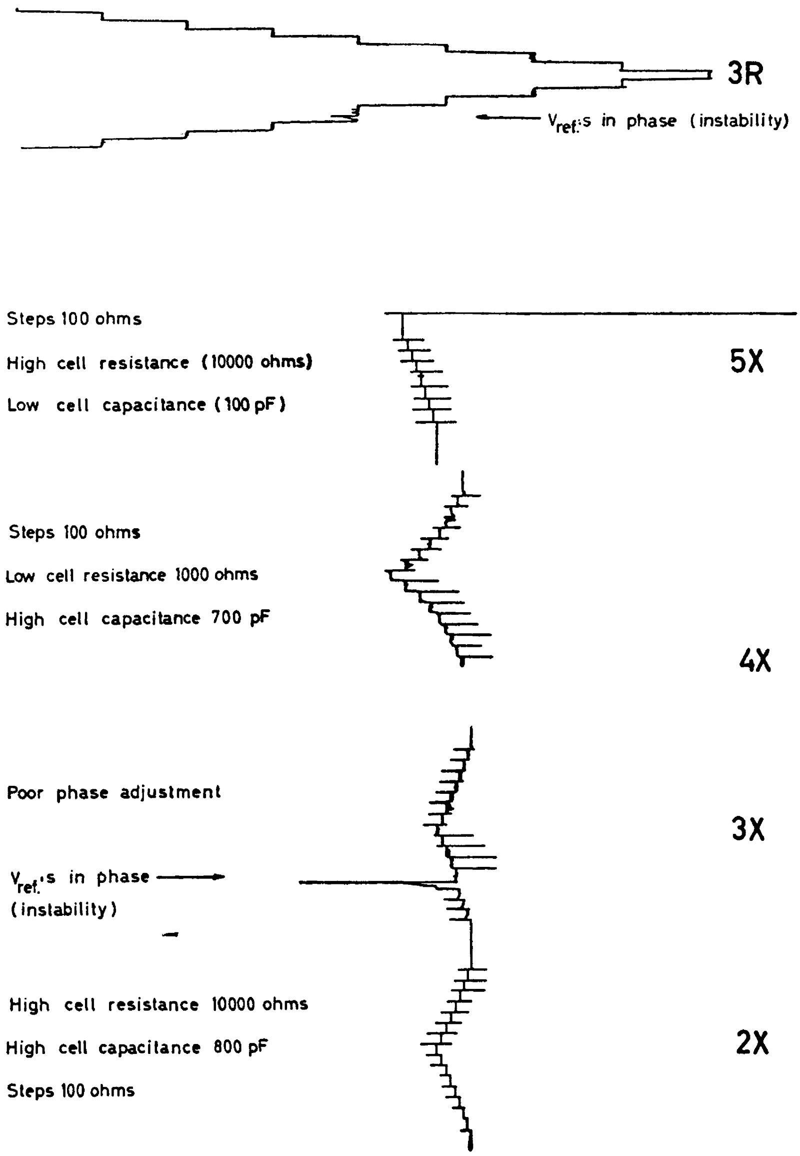

Fig. 18. Double servo-recordings using the bridge shown in Fig. 1. Square wave reference potentials.

The servo-controlled precision resistance was a wire-wound linear resistance of 1 000 ohms (*cf.* Ref.[1]), and as "cell" was used a Leeds & Northrup six-decade resistance (0 to 11 000 ohms in steps of 0.01 ohm) in parallel with a precision-variable-condenser covering the range 100 to 1 000 pF (General Radio Co., type N).

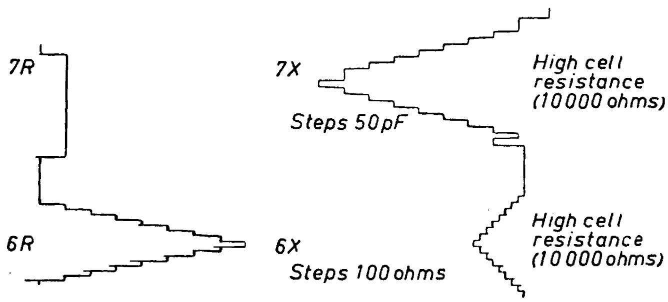

Fig. 19. Double servo-recordings using potentiometer bridge shown in Fig. 7. No shunt used. Square wave reference potentials.

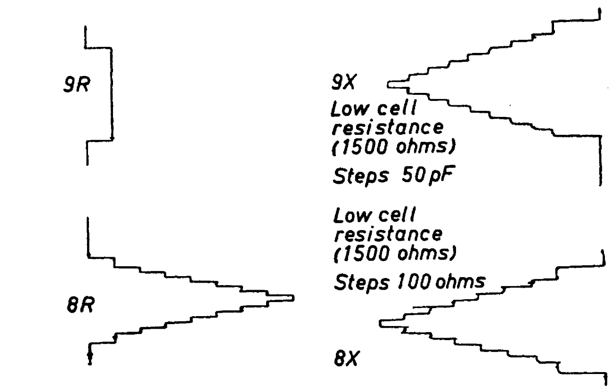

Fig. 20. Double servo-recordings using potentiometer bridge shown in Fig. 7. No shunt used. Square wave reference potentials.

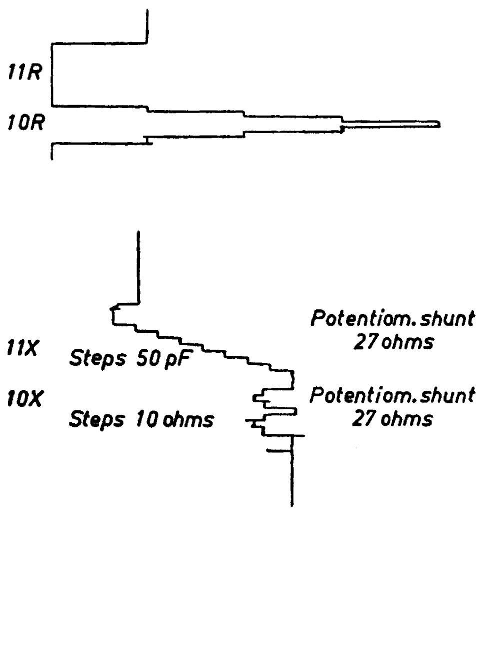

Fig. 21. Double servo-recordings using potentiometer bridge shown in Fig. 7. Recorder range 50 ohms (potentiometer shunted by 27 ohms).

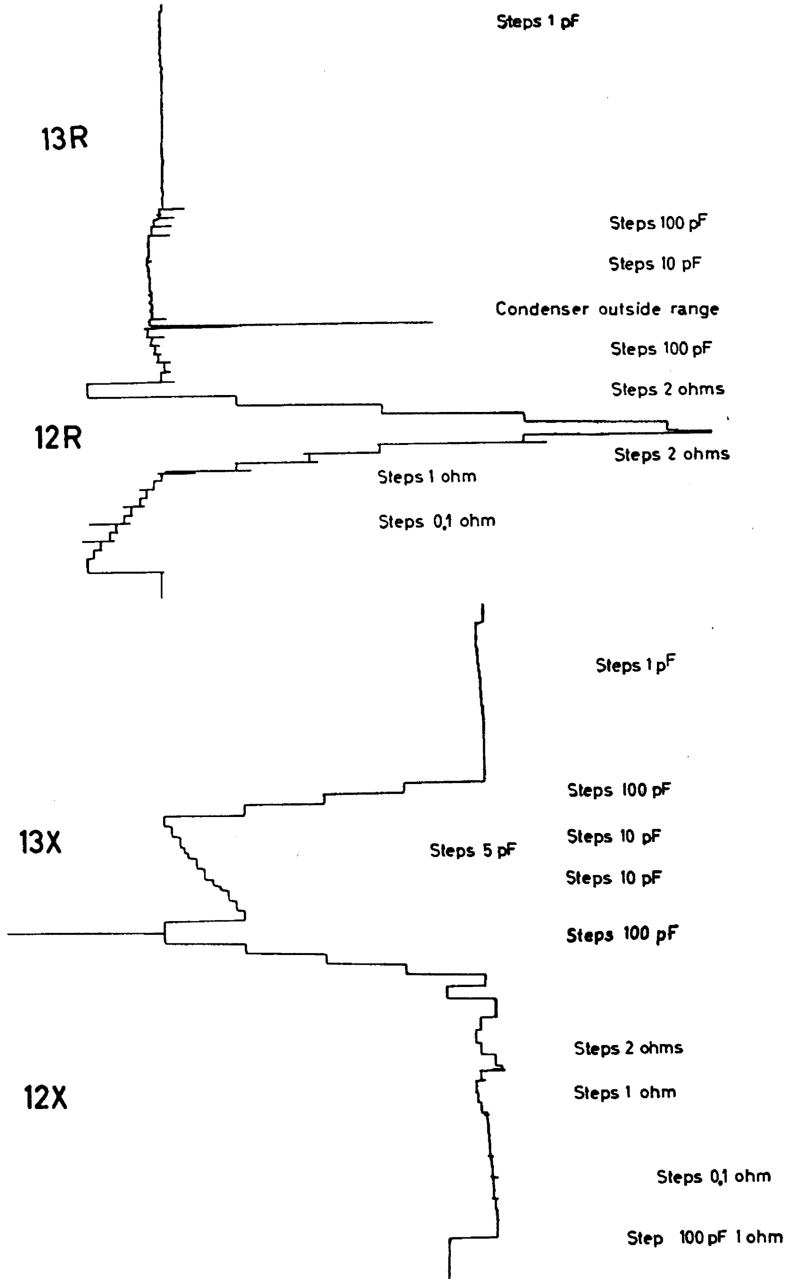

Fig. 22. Double servo-recordings using potentiometer bridge shown in Fig. 7. Recorder range 10 ohms (low resistance shunt).

The adjustment of the reference potential was found difficult, however, because operation of the phase shifter also gave a change in amplitude of the reference signal. As the two mixer-tubes in the homodyne rectifier did not have absolutely identical characteristics, this amplitude change led to an output signal from the detector even in the absence of an input signal from the bridge. Best results were obtained when the phases of the reference potentials were adjusted in such a way that small variations in resistance gave maximum signal to the resistance servo, and small variations in capacitance maximum signal to the capacitance servo. It was found later that perfect phasing could be obtained by allowing the sinusoidal reference potentials to trigger a square-wave generator (Philips type GM 2314) which delivered a square-wave reference potential of constant amplitude (pulse height) to the third grid of the L 7 mixer tubes in the synchronous rectifier. The initial adjustments could then be made as outlined in section I.

The servos were comparatively tolerant with respect to smaller degrees of incorrect phasing of the reference signals, especially at high values for "cell" resistance.

The resistance recorder was completely unaffected by changes in capacitance, but the capacitance recorder was sensitive to resistance changes, probably because the wire-wound, 1 000 ohms measuring resistance in the resistance recorder had a higher residual inductance than the "cell" resistance. The recordings $1R$, $1X$, $2X$, $3R$, $3X$, $4X$, and $5X$ shown in Figs. 17 and 18 were obtained with the bridge connected according to Fig. 1. $1X$ and $1R$ were obtained with sinusoidal reference signals, and show that the system remains stable even when the resistance and capacitance to be measured are changed in a step-wise manner, either one at a time or both simultaneously.

Compared with $2X$, $3X$ shows the influence of poor phasing. $3X$ and $3R$ show that the system becomes unstable when the reference signals are in phase. Compared with $2X$, $4X$ shows how the influence of resistance-changes on the capacitance recorder increases with decreasing total "cell" resistance. This influence did not tend to increase with decreasing total "cell" capacitance, however, as indicated by $5X$ compared with $2X$.

When the bridge was connected according to Fig. 7 with the measuring condenser connected in parallel with X_1 the influence of resistance changes on the capacitance recorder increased, as a comparison between the recordings $6X$ and $8X$ of Figs. 19 and 20, with $2X$ and $4X$, respectively, shows. With the circuit of Fig. 7, change in resistance introduces an extra reactive unbalance over and above that caused by the inductance of the measuring resistance. The resistance recorder is unaffected by capacitance changes, as shown by the recordings $7R$ and $9R$ of Figs. 19 and 20, and the R scale is completely linear from $R = 1\,000$ ohms to $R = 10\,000$ ohms ($6R$ and $8R$).

Shunting of the potentiometer caused a marked change in the influence of resistance changes on the capacitance recordings. At a measuring range of 50 ohms (shunt of 27 ohms), the capacitance recorder changed direction when the mid-point of the potentiometer was passed, as shown by the recordings $10X$ and $12X$ of Figs. 21 and 22, respectively. As shown by the recording $11R$ (Fig. 21) the resistance recorder was still insensitive to capacitance changes; but when the resistance range was decreased to 10 ohms, a certain influence became apparent ($13R$ of Fig. 22), owing to the fact that the measuring condenser across X_1 in Fig. 7 did not have exactly the same influence as X_C in the circuit. This source of error can be eliminated by using the circuit shown in Fig. 8.

CONCLUSIONS

a) It is desirable to use square-wave reference signals of constant amplitude in the homodyne rectifier, and it seems likely that the best type of detector would be a balanced push-pull detector working with square-wave input and reference signals, as discussed by Farren.[3]

b) A completely non-inductive measuring resistance in the resistance servo should be used in order to avoid errors of the capacitance recorder.

c) It is desirable to use the double servo system even in cases where only one component is being recorded, in order to preserve bridge balance. Single servo operation has been discussed in section V.

d) For direct recording of parallel components (resistance and capacitance), the bridge circuit of Fig. 1 is preferable on theoretical grounds. The potentiometer bridge of Fig. 8 may be used when φ is small or resistance only is to be recorded.

APPENDIX I

Calculation of α_R and α_X

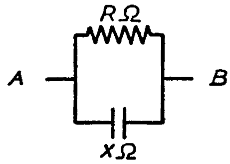

Fig. 23. Resistance and reactance in parallel.

Let us put the total impedance AB of the parallel combination of the resistance R and condenser X of Fig. 23 equal to Z. We then have $\frac{1}{\overline{Z}} = \frac{1}{R} + \frac{1}{jX}$ *;

$$\overline{Z} = \frac{RjX}{R + jX} = \frac{RjX\,(R - jX)}{R^2 + X^2} = \frac{RX^2 + R^2jX}{R^2 + X^2} \tag{1}$$

$$\text{tg } \varphi = \frac{R}{X} \tag{2}$$

We want to know how $\overline{Z}$ changes when X is kept constant and R varies, and vice versa. $\overline{Z}$ forms the angle φ with the current vector, and we assume that $\frac{\partial \overline{Z}}{\partial R}$ and $\frac{\partial \overline{Z}}{\partial X}$ form the angles γ_R and γ_X, respectively, with the current vector.

$$\frac{\partial \overline{Z}}{\partial R} = \frac{(R^2 + X^2)\ (X^2 + 2\ RjX) - 2R(RX^2 + R^2jX)}{(R^2 + X^2)^2} =$$

$$= (X^2 - R^2 + 2RjX)\ \frac{X^2}{(R^2 + X^2)^2} \tag{3}$$

$$\text{tg } \gamma_R = \frac{2RX}{X^2 - R^2} = \frac{2\,\frac{R}{X}}{1 - \frac{R^2}{X^2}} = \frac{2 \text{ tg } \varphi}{1 - \text{tg}^2\varphi} \tag{4}$$

$$\gamma_R = 2\ \varphi \tag{5}$$

$$\frac{\partial \overline{Z}}{\partial X} = \frac{(R^2 + X^2)\ (RX + j2R^2) - 2X(RX^2 + R^2jX}{(R^2 + X^2)} =$$

$$= \left[2RX + j(R^2 - X^2)\right] \frac{R^2}{(R^2 + X^2)^2} \tag{6}$$

* A good account of symbolic alternating current theory is given by Kemp [4].

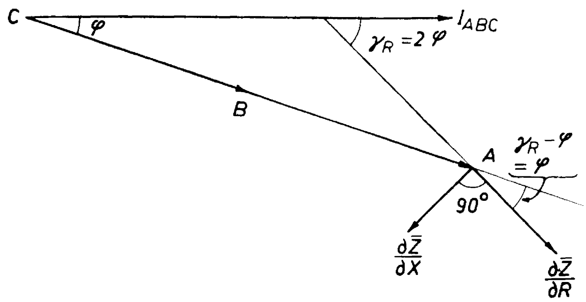

Fig. 24. Diagram showing the directions of certain partial differential quotients related to the bridge shown in Fig. 1. For explanation see text.

$$\text{tg}\ \gamma_X = \frac{R^2 - X^2}{2\ RX} = \frac{\frac{R^2}{X^2} - 1}{2\frac{R}{X}} = \frac{\text{tg}^2\varphi - 1}{2\ \text{tg}\ \varphi} = -\frac{1}{\text{tg}\ \gamma_R} \tag{7}$$

$$\gamma_X = \gamma_R + 90° \tag{8}$$

If the current I_{BD} (Fig. 1) through the input circuit of the amplifier can be neglected, the current I_{AB} will be the same as the current $I_{BC} = I_{ABC}$. Fig. 24 shows this current and the impedances AB ($= \bar{Z}$) and BC at balance. If R or X are changed, the point A will move in the direction of $\partial\bar{Z}/\partial R$ or $\partial\bar{Z}/\partial X$ (respectively). If the current I_{ABC} were constant we could let the impedance vectors represent the potential vectors as well, because the latter would have the same length and direction as the impedance vectors relative to the current I_{ABC}. However, I_{ABC} does not remain constant; but the potential V_{CA} is constant provided that the oscillator has a low output resistance. If we let the impedance vectors also represent the potential vectors and change the position of point A in the diagram (*i.e.* alter $\bar{Z}$), we must therefore let the vectors stretch or shrink until the distance CA becomes the same as before, if they are to represent the potentials after the change. In Fig. 25 the point A has moved to A′ in the impedance diagram, owing to an increase of the reactance X. The vectors $\overline{CB}$, $\overline{BA}$, and $\overline{CA}$ represent the impedances and the potentials before the change. The vectors $\overline{CB}$, $\overline{BA}'$, and $\overline{CA}'$ represent the impedances after the change. In order to obtain the potential-vectors after the change, we stretch $\overline{CA}'$ to $\overline{CA}'_1$ which is equal in length to $\overline{CA}$. We then draw the line BA′ and a line parallel to BA′ through A_1'. The latter cuts CA at point B_1. The vectors $\overline{CB}_1$, $\overline{B_1A_1}'$, and $\overline{CA}_1'$ then represent the potentials, because they are related to each other in the same manner as the impedance vectors and furthermore are parallel to the latter, and CA_1' is equal to CA. We now denote the mid-points of CA′ and CA_1' by B′ and B_1'. So far the increase in X has meant that in the potential diagram the point B moves

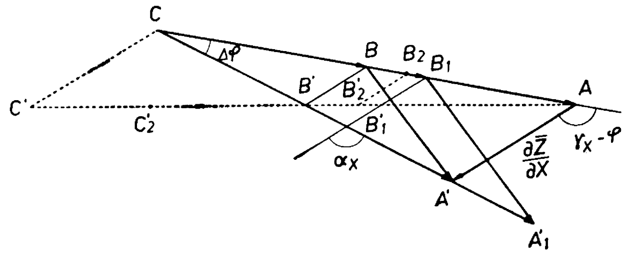

Fig. 25. Geometrical derivation of α_R and α_X of Fig. 3. For explanation see text.

to B_1 and A to A_1'. We may as well consider $\overline{CA_1'}$ to represent the initial potential across AC, however. In this case the change in $\overline{X}$ makes point B move from B_1' to B_1. The triangle $A_1'B_1C$ of Fig. 25 is then similar to the triangle ABC of Fig. 2, and, since only X has been changed, the line $B_1'B_1$ corresponds to l_X. $B_1'B_1$ is parallel to BB' because they occupy the same position in the two similarly positioned triangles A'BC and $A_1'B_1C$. The line BB' is parallel transverse of AA'C, since it cuts the sides AC and A'C at their mid-points. B'B is thus parallel to AA'. If $\Delta\ \varphi$ is small, CA_1' will be parallel to CA, and it follows that $\alpha_X = \gamma_X - \varphi = \varphi + 90°$. Similarly, it may be shown that $\alpha_R = \gamma_R - \varphi = \varphi$.

If instead of increasing X we had decreased X_C (Fig. 1), point C in the diagram of Fig. 25 would have moved to C'. B' would have remained unchanged, but instead of A_1', B_1, and B_1' we would have obtained C_2', B_2, B_2'. B_2B' is parallel to CC' and AA'. When CC' ($= \Delta\ \overline{Z}$) $\rightarrow 0$, $B_2B'_2 \rightarrow l_X$ and $\alpha_X \rightarrow \gamma_X - \varphi$. The line l_X thus remains the same in both cases.

APPENDIX II

The quantitative relations between the measured and the servo-operated impedance components in the case of the "potentiometer" bridge arrangement

The impedance diagram for the bridge arrangement of Fig. 8 is shown in Fig. 9. The lines l_{R_e}, l_{R_C}, and l_X have been drawn in through point B although strictly they belong to the potential diagram. It can be seen directly that for the circuit of Fig. 8 l_{R_e} is parallel to I_{ABC}, whereas l_{R_C} and l_X form the angles 2 φ_2 and 2 $\varphi_2 + 90°$ with the current I_{ABC}. In the impedance diagram of Fig. 26 we assume that R_e is small, so that $\overline{Z} = \overline{Z}_2$ and $\varphi = \varphi_2$. If R_C is now changed by $\Delta\ R_C$, the point C will move the distance $\vartheta\ R_C$ to C' towards $\overline{\partial Z/\partial R_C}$. The bridge is now to be balanced, and for the purpose of discussion we may assume that the servos operate one at a time (in practice they work simultaneously). The R_e-servo operates first. R_{e1} decreases and R_{e2} increases simultaneously by the same amount, which means that the points A and C' in Fig. 26 move at the same rate and in the same direction, along a path parallel to I_{ABC}. It should be observed that point B retains its position, and will do so in the following because we are working with the impedance diagram

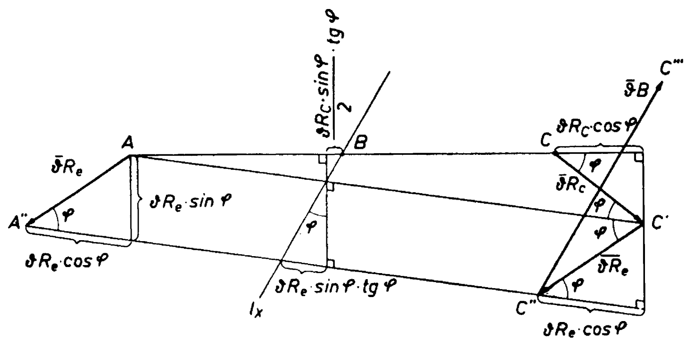

Fig. 26. Vector diagram showing the balancing of the bridge shown in Fig. 8. For explanation see text.

and not with the potential diagram. AC will thus be displaced parallel to itself. The R_e servo will stop when the line DB in the potential diagram (Fig. 2) coincides with l_X. The potential triangle will be conformate to the impedance triangle. The servo thus stops when the mid-point of AC reaches l_X in the impedance diagram. AC then occupies the position A″C″. Now the X-servo starts, and moves point B in the potential triangle (Fig. 2) along l_X to C. In the impedance diagram of Fig. 26 this corresponds to a movement of C″ towards "north-east" along a line parallel to l_X to a point C‴ on line A″B prolonged. The result of these operations is, of course, that both in the potential and the impedance diagrams the triangle ABC turns into a straight line with point B at the mid-point.

As we work with condensers that have a linear capacitance variation we introduce the susceptance $B_2 = 1/X_2$ instead of X_2. We shall now investigate how R_{e2} and B_2 have to be changed in order to return the bridge to balance when R_C is changed by ΔR_C. When R_C is changed by the amount ΔR_C, $\overline{Z}$ will be changed by the amount $\partial\overline{Z}/\partial R_C \cdot \Delta\, R_C = \overline{\vartheta}\, R_C$. In order to compensate this, R_{e1} is first altered by ΔR_{e1} (and R_{e2} by $\Delta R_{e2} = -\Delta R_{e1}$), whereby $\overline{Z}$ is changed by $\overline{\vartheta}\, R_{e1}$. Finally, B_2 is changed by ΔB, and $\overline{Z}$ by $\overline{\vartheta}$ B. $\vartheta\, R_e$, ϑ R_C, and ϑ B are indicated in Fig. 26. Bearing in mind that AC is very large compared with the differentials, and that A″C″ is going to be cut at the mid-point by l_X (so that we must add the same amount to both sides of l_X when AC is displaced from AC to A″C″) we obtain (Fig. 21) $\vartheta R_C \cos\varphi + \frac{1}{2}\vartheta\, R_C \sin\varphi\,\mathrm{tg}\,\varphi + \vartheta R_e \sin\varphi\,\mathrm{tg}\,\varphi - \vartheta R_e \cos\varphi = \vartheta\, R_e \cos\varphi - \frac{1}{2}\vartheta\, R_C \sin\varphi\,\mathrm{tg}\,\varphi - \vartheta R_e \sin\varphi\,\mathrm{tg}\,\varphi$;

$$\vartheta R_e = \frac{1 + \mathrm{tg}^2\,\varphi}{1 - \mathrm{tg}^2\,\varphi} \cdot \frac{1}{2}\,\vartheta\, R_C \qquad (9)$$

We now turn to ϑB. The point C is evidently going to be displaced from the point C″ (at a distance of ($\vartheta R_C \sin\varphi + \vartheta R_e \sin\varphi$) from the line AC) to a point C‴ (at a distance of $\vartheta R_e \sin\varphi$ on the other side of AC). The displacement has to take place in a direction that forms angle φ with the normal to AC. The distance ϑ B will be,

$$\vartheta\ \mathrm{B} = \frac{2\ \vartheta\ R_e \sin\ \varphi + \vartheta\ R_C \sin\ \varphi}{\cos\ \varphi} = (2\ \vartheta Re + \vartheta\ R_c) \mathrm{tg}\ \varphi =$$

$$= \vartheta\ R_C \mathrm{tg}\ \varphi\ \left(1 + \frac{1 + \mathrm{tg}^2\ \varphi}{1 - \mathrm{tg}^2\ \varphi}\right).$$

$$\vartheta\ \mathrm{B} = \frac{2\ \mathrm{tg}\ \varphi}{1 - \mathrm{tg}^2\ \varphi} \cdot R_C \qquad (10)$$

$$\overline{\vartheta}\ R_C = \overline{\frac{\partial\ Z}{\partial\ R_C}} \cdot \Delta\ R_C;\ \vartheta\ R_C = \frac{\partial\ Z}{\partial\ R_C} \cdot \Delta\ R_C$$

$\frac{\partial\ Z}{\partial\ R_C}$ is obtained from $\overline{\frac{\partial\ Z}{\partial\ R_C}}$: which has been calculated previously, eqns. 2 and 3. We obtain $\frac{\partial\ Z}{\partial\ R_C} = \frac{1}{1 + \mathrm{tg}^2\ \varphi}$, and

$$\vartheta\ R_C = \frac{1}{1 + \mathrm{tg}^2\ \varphi} \cdot \Delta\ R_C \qquad (11)$$

$\frac{\partial\ Z}{\partial\ X}$ is obtained from $\overline{\frac{\partial\ Z}{\partial\ X}}$, and is equal to $\frac{\mathrm{tg}^2\ \varphi}{1 + \mathrm{tg}^2\ \varphi}$

$$\frac{\partial\ Z}{\partial\ \mathrm{B}} = -\frac{1}{\mathrm{B}^2} \cdot \frac{\partial\ Z}{\partial\ X} = -\frac{R_C}{\mathrm{tg}^2\ \varphi} \cdot \frac{\partial\ Z}{\partial\ X} = -\frac{R_C^2}{1 + \mathrm{tg}^2\ \varphi} \qquad (12)$$

(Both X_C and X_2 are included in X and B).

$$\Delta\ \mathrm{B} = \frac{\vartheta\ \mathrm{B}}{\frac{\partial\ Z}{\partial\ \mathrm{B}}} = \frac{2\ \mathrm{tg}\ \varphi}{1 - \mathrm{tg}^2\ \varphi} \cdot \frac{1}{1 + \mathrm{tg}^2\ \varphi} \cdot \frac{1 + \mathrm{tg}^2\ \varphi}{-R_C^2} \cdot \Delta\ R_C$$

$$\Delta \mathrm{B}_2 = -\frac{2\ \mathrm{tg}\ \varphi}{R_C^2\ (1 - \mathrm{tg}^2\ \varphi)} \cdot \Delta\ R_C \qquad (13)$$

R_{e2} forms a part of the series component of the impedance Z, and we get

$$\Delta R_e = \vartheta\ R_e = \frac{1 + \mathrm{tg}^2\ \varphi}{1 - \mathrm{tg}^2\ \varphi} \cdot \frac{1}{2} \cdot \frac{1}{1 + \mathrm{tg}^2\ \varphi} \cdot \Delta\ R_C$$

$$\Delta R_e = \frac{1}{1 - \mathrm{tg}^2\ \varphi} \cdot \frac{1}{2} \cdot \Delta\ R_C \qquad (14)$$

For the sake of completeness we also record how B_2 is related to X_C

$$\Delta \mathrm{B}_2 = -\Delta \frac{1}{X_C} \qquad (15)$$

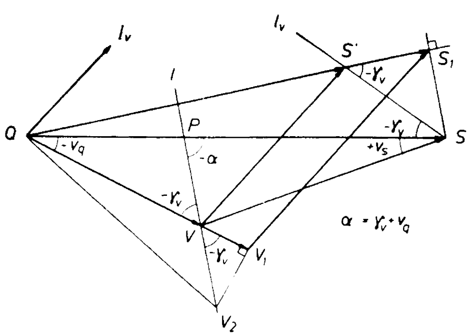

Fig. 27. Geometrical derivation of a_R and a_G of the Z—Y bridge. For explanation see text.

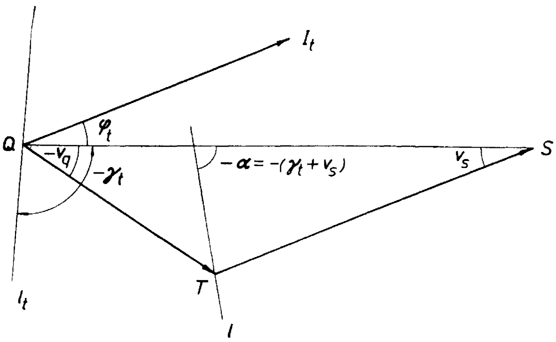

Fig. 28. Geometrical derivation of a_R and a_G of the Z—Y bridge. For explanation see text.

APPENDIX III

Calculations of a_G and a_R of the ZY-bridge

Looking at Fig. 27 we first assume that the bridge is balanced. We let the triangle QSV represent both potentials and impedances. The impedance VS is now altered, causing the point S to move along a line l_v (forming the angle γ_v with QS) to S′. The triangle QS′V represents the impedance after the unbalance has been introduced. In order to get the potential triangle we enlarge the triangle QS′V to QS_1' V_1 (sp that $QS_1' = QS$) and rotate it about Q to position QSV_2. We assume that angle $S_1'QS$ is very small, and that QS_1' is therefore parallel to QS. Triangles $S'S_1'S$ and VV_1V_2 are then conformate (proportional catheti), and the angle V_1VV_2 is equal to $-\gamma_v$. From the triangle QPV we get $\alpha = \gamma_v + v_q$, where α is the angle between line l, along which point V is displaced, and QS. When the change in the impedance VS is purely reactive, l_v will be normal to the current I_v in QVS. The impedance QV is purely reactive, and is therefore also normal to I_v. We therefore obtain $\gamma_v = v_q$ and $\alpha = 2\ v_q$. If the change in VS is purely resistive, l_v, and therefore also l, will be normal to the previous position. Similarly, for the triangle QST (Fig. 28) we get $\alpha = \gamma_t + v_s$, when Q moves along the line l_t in a direction forming angle γ_t with QS.

The impedance QT (Fig. 28) consists of a resistance and a reactance in parallel. When only the resistive component is changed we have previously shown (see Appendix I) that the line of displacement l_t of the point Q forms an angle 2 φ_t with the current. φ_t is the angle between the impedance QT and the current vector I_t. This gives $\varphi_t - \gamma_t = 2\ (\varphi_t - v_q)$ or $\varphi_t = 2\ v_q - \gamma_t$. The impedance ST is purely resistive, and is therefore parallel to I_t. Therefore $v_s = \varphi_t = 2\ v_q - \gamma_t$, which gives $\alpha = \gamma_t + v_s = \gamma_t + 2\ v_q - \gamma_t$, or $\alpha = 2\ v_q$. If the reactive component only of the impedance QT is changed we obtain lines l_t and l normal to those in the case just dealt with.

It follows from the discussion above that a change in the reactive component of the impedance SV or in the conductance QT displaces the points V and T, respectively in the potential diagram along the same line l_G, and may therefore be used to compensate each other. Similarly, a change in the resistance of SV or the susceptance of QT will displace V or T along a line l_R normal to l_G. The angles α_G and α_R formed by the lines with the voltage QS are,

$$\alpha_G = 2v_q \quad (16)$$

$$\alpha_R = 2v_q + 90° \quad (17)$$

Acknowledgements. The theory given in this paper has been worked out by F. Möhl. We are indebted to Svenska AB Philips for the loan of square-wave generators, to Seve Radio, Mölnlycke, for the construction of apparatus, and to Professor Olof Mellander for laboratory facilities.

This work was made possible by a grant from the *Swedish Medical Research Council* to one of us (E.S.).

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Received November 8, 1957.